# CHAPTER 12

# Composition of Simple Harmonic Motions

## 12.1. COMPOSITION OF TWO SIMPLE HARMONIC MOTIONS IN A STRAIGHT LINE

The two simple harmonic vibrations are represented by the equations

$$y_1 = a_1 \sin(\omega t + \alpha_1) \quad \text{...(1)}$$

and

$$y_2 = a_2 \sin(\omega t + \alpha_2) \quad \text{...(2)}$$

$y_1$ and $y_2$ are the displacements of a particle due to the two vibrations.

$a_1$ and $a_2$ are the amplitudes of the two vibrations.

$\alpha_1$ and $\alpha_2$ are the epoch angles.

The angular frequency $\omega$ is the same for both vibrations. The resultant displacement $y$ of the particle is given by,

$$\begin{aligned} y &= y_1 + y_2 \\ &= a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2) \\ &= a_1 (\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + \\ &\quad a_2 (\sin \omega t \cos \alpha_2 + \cos \omega t \sin \alpha_2) \end{aligned}$$

$$y = (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin \omega t + (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos \omega t \quad \text{...(3)}$$

We can write the displacement of the resultant motion as

$$y = A \sin(\omega t + \phi)$$

$$y = A \sin \omega t \cos \phi + A \cos \omega t \sin \phi \quad \text{...(4)}$$

$A$ is the amplitude of the resultant motion and $\phi$ is the initial phase.

Equating the coefficients of $\sin \omega t$ and $\cos \omega t$ in Eqs. (3) and (4),

$$A \cos \phi = \alpha_1 \cos \alpha_1 + a_2 \cos \alpha_2 \quad \text{...(5)}$$

$$A \sin \phi = a_1 \sin \alpha_1 + a_2 \sin \alpha_2 \quad \text{...(6)}$$

Squaring Eqs. (5) and (6) and adding,

$$A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\alpha_1 - \alpha_2) \quad \text{...(7)}$$

Dividing equation (6) by (5),

$$\tan \phi = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2} \quad \text{...(8)}$$

Equations (7) and (8) give the values of $A$ and $\phi$.

## 12.2. COMPOSITION OF TWO SIMPLE HARMONIC MOTIONS OF EQUAL TIME PERIODS AT RIGHT ANGLES

Suppose the same particle is simultaneously subjected to two S.H. motions along two mutually perpendicular directions ($X$ and $Y$ axes). Let the displacements of the S.H. motions be

$$x = a \sin(\omega t + \alpha) \quad ...(1)$$

and

$$y = b \sin \omega t \quad ...(2)$$

$\alpha$ = The phase difference betwean the two S.H. motions.

The resultant motion is obtained by eliminating $t$ from (1) and (2).

From equation (2),

$$\sin \omega t = \frac{y}{b}$$

$$\therefore \quad \cos \omega t = \sqrt{1 - \frac{y^2}{b^2}}$$

From equation (1), $\frac{x}{a} = [\sin \omega t \cos \alpha + \cos \omega t \sin\alpha]$ ...(3)

Substituting the values of $\sin \omega t$ and $\cos \omega t$ in equation (3),

$$\frac{x}{a} = \left[\frac{y}{b}\cos\alpha + \sqrt{1 - \frac{y^2}{b^2}}.\sin\alpha\right]$$

$$\text{or} \quad \frac{x}{a} - \frac{y}{b}\cos\alpha = \left(\sqrt{1 - \frac{y^2}{b^2}}\right)\sin\alpha$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\alpha - \frac{2xy}{ab}\cos\alpha = \left(1 - \frac{y^2}{b^2}\right)\sin^2\alpha$$

$$\text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2}[\sin^2\alpha + \cos^2\alpha] - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha \quad ...(4)$$

**This represents the general equation of an ellipse.**

**Hence the resultant motion is in general elliptical.**

**Special cases:**

(*i*) If $\alpha = 0$, $\cos\alpha = 1$; $\sin\alpha = 0$

$$\frac{x}{a} - \frac{y}{b} = 0$$

$$\text{or} \quad y = \frac{b}{a}x.$$

This represents a straight line $AB$ (Fig. 12.1) of slope $b/a$.

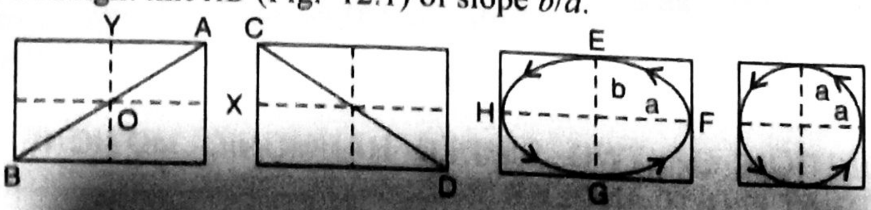



Fig. 12.1

(*ii*) If $\alpha = \pi$, $\sin\alpha = 0$; $\cos\alpha = -1$; Hence $y = -\frac{b}{a}x$.

The resultant motion is represented by the straight line *CD* passing through the origin and having a negative slope.

(*iii*) If $\alpha = \frac{\pi}{2}, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

This represents an ellipse *EFGH* whose major and minor axes coincide with the coordinate axes.

(*iv*) If $\alpha = \pi/2$ and $a = b$, then $x^2 + y^2 = a^2$.

This represents a circle of radius *a*. Hence, the resultant motion is circular.

## Lissajous Figures

When a particle is acted upon simultaneously by two S.H. motions at right angles to each other, the resultant path traced by the particle is called *Lissajous figures*. The shape of the Lissajous' Figure depends upon (*i*) the amplitudes, (*ii*) the periods of the two component motions and (*iii*) the phase difference between them.

## 12.3. EXPERIMENTAL METHODS FOR OBTAINING LISSAJOUS' FIGURES

**1. Optical method.**

*A* and *B* are two tuning forks with frequencies in the ratio of 2 : 1 (Fig. 12.2).

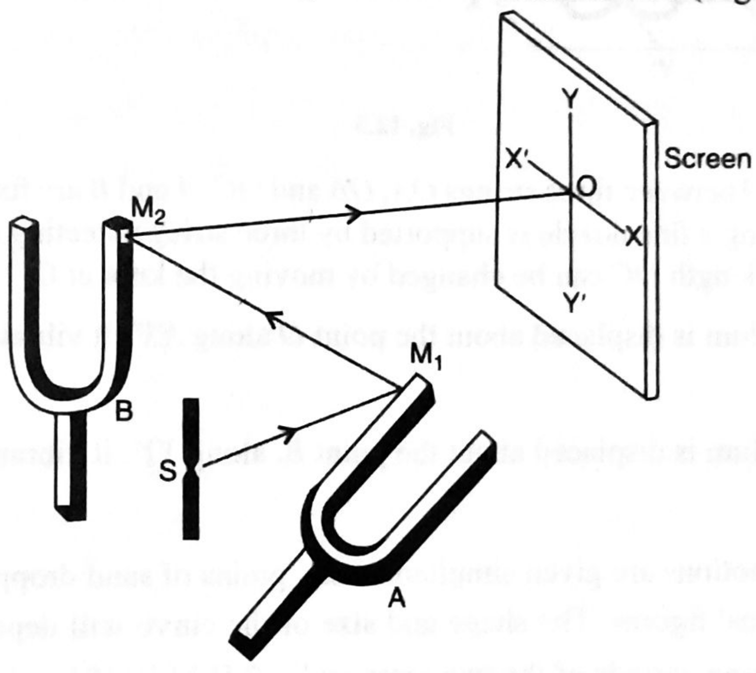


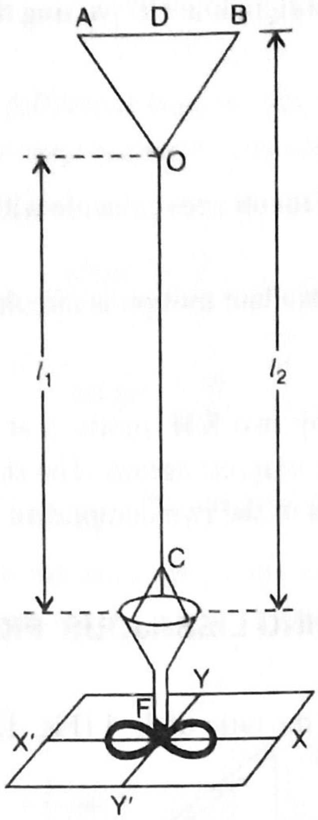

**Fig. 12.2**

The prongs of A vibrate in a horizontal plane. The prongs of *B* vibrate in a vertical plane. $M_1$ and $M_2$ are two small plane mirrors attached to the prongs of *A* and *B* respectively. It is adjusted that the spot of light after reflection form $M_1$ and $M_2$ is obtained at *O*, the centre of the screen.

- When only the tuning fork *A* vibrates, the spot of light moves along *XX'*.
- When only the tuning fork *B* vibrates, the spot of light moves along *YY'*.
- When *A* and *B* vibrate simultaneously and are in phase, the spot of light traces the figure of eigtht.

**2. Blackburn's Pendulum**

Fig. 12.3 shows a simple form of this pendulum.



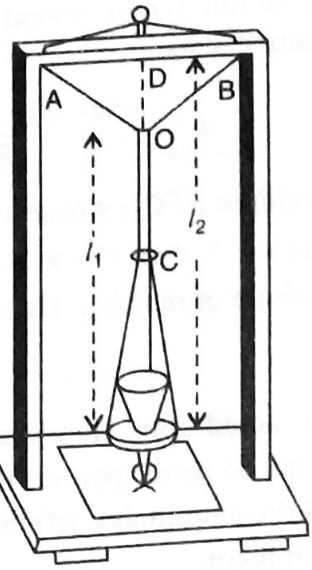




Fig. 12.3

A knot is made at $O$ between three strings $OA$, $OB$ and $OC$. $A$ and $B$ are fixed to hooks in a rigid frame. A funnel ($F$) having a fine nozzle is supported by three strings meeting at $C$. Dry fine sand is taken in the funnel. The length $OC$ can be changed by moving the knot at $O$.

- When the pendulum is displaced about the point $O$ along $XX'$, it vibrates with a time period $t_1 = 2\pi\sqrt{l_1 / g}$.
- When the pendulum is displaced about the point $B$, along $YY'$, it vibrates with a time period $t_2 = 2\pi\sqrt{l_2 / g}$.
- When both the motions are given simultaneously, grains of sand dropping out of the funnel trace out Lissajous' figures. The shape and size of the curve will depend upon the relative amplitudes and time periods of the two rectangular S.H.M.'s. If $l_1 = l_2/4$, then $t_1/t_2 = \sqrt{1/4}$ $= 1/2$, giving us a figure of '8'. If the ratio between time periods is not exactly 1:2, the resultant motion will gradually pass through various phases shown in Fig. 12.1.

## 12.4. USES OF LISSAJOUS FIGURES

(1) The form of a Lissajous figure enables us to determine the ratio of the time periods of component vibrations. From the Lissajous' figure, the number of times the curve touches the horizontal and the vertical sides of a rectangle bounding the Lissajous figure, is found. If the curve touches the horizontal side $p$ times and the vertical side $q$ times, the ratio of the time periods is $p : q$.

(2) The Lissajous figures are used for the accurate determination of the frequency of a tuning fork.

(3) Lissajous figures are used in obtaining beautiful designs for calico printing in textile industry.

# 7 CHAPTER Moment of Inertia

## 7.1. INTRODUCTION

**Definition** : If a rigid body consists of a finite number of particles of masses $m_1$, $m_2$, $m_3$ etc., at distances $r_1$, $r_2$, $r_3$, etc., from a given straight line $XY$ (Fig. 7.1), the moment of inertia of the body about the given line is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ....... = \Sigma m r^2$$

**Unit** : $kgm^2$;

Dimensional formula: $[I] = [ML^2]$

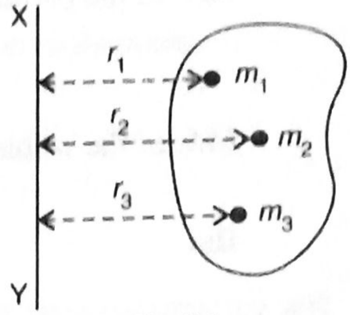


Fig. 7.1

**Radius of Gyration** : Suppose the whole mass of the body ($M$) is concentrated at a point distant $k$ from the axis such that $M.k^2 = \Sigma m r^2 = I$. Then $k$ is called the radius of gyration of the body about the given axis. $k = \sqrt{I/M}$.

**Physical Significance of M.I.** In translational motion $F = m a$.

In rotational motion $\vec{\tau} = I \alpha$. This suggests that just as we associate a force with the linear acceleration of a body, so we may associate a torque with the angular acceleration of a body about a given axis. Mass $M$ is a measure of the resistance a body offers to having its translational motion changed by a given force. Similarly, moment of inertia $I$ is a measure of the resistance a body offers to having its rotational motion changed by a given torque. Thus *M.I. plays the same role in rotational motion as mass does in translational motion.*

## 7.2. PERPENDICULAR AXES THEOREM

**Statement** : *If $I_x$ and $I_y$ are the moments of inertia of a lamina about two rectangular axes OX and OY in its plane, its moment of inertia about an axis OZ, perpendicular to its plane, is $I_z = I_x + I_y$*

**Proof** : Let $OX$, $OY$ be the two perpendicular axes in the plane of the lamina and $OZ$ an axis perpendicular to the lamina (Fig. 7.2). Consider a particle $P$, of mass $m$ in the plane of the lamina. $x$, $y$ and $r$ are the distances of the particle from $OY$, $OX$ and $OZ$ respectively.

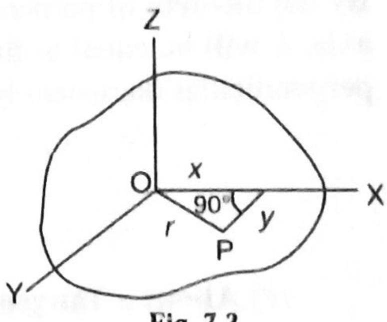


Fig. 7.2

Moment of inertia of the particle about $OZ = m r^2$

$\therefore$ M.I. of the lamina about $OZ = \Sigma m r^2$

Similarly, M.I. of the lamina about $OX = \Sigma m y^2$

and M.I. of the lamina about $OY = \Sigma m x^2$

But $r^2 = x^2 + y^2$

$\therefore$ $\Sigma m r^2 = \Sigma m (x^2 + y^2) = \Sigma m x^2 + \Sigma m y^2$

$\Sigma m r^2 = I_z$; $\Sigma m x^2 = I_y$; $\Sigma m y^2 = I_x$

$\therefore$ $I_z = I_x + I_y$

## 7.3. THEOREM OF PARALLEL AXES

**Statement :** *If $I$ is the moment of inertia of a body about an axis through its centre of mass and $I'$ its moment of inertia about a parallel axis at a perpendicular distance $h$ from the first axis, then $I' = I + M h^2$, where $M$ is the mass of the body.*

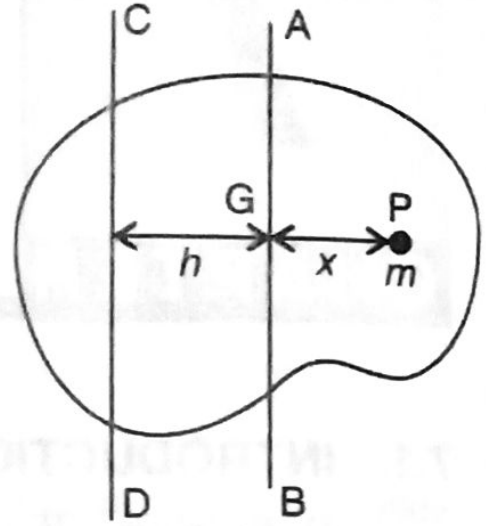


**Fig. 7.3**

**Proof** : $AB$ is an axis passing through the centre of mass of the body $G$ (Fig. 7.3.) $CD$ is a parallel axis at a perpendicular distance $h$ from $AB$. $M$ is the mass of the body.

Consider a particle $P$ of mass $m$ at a distance $x$ from $AB$.

M.I. of the particle $P$ about $AB = mx^2$

M.I. of the whole body about $AB = I = \Sigma m x^2$

M.I. of the particle $P$ about $CD = m(x + h)^2$

$$= m (x^2 + h^2 + 2 x h)$$

$$= mx^2 + m h^2 + 2 m x h.$$

M.I. of the whole body about $CD$

$$= I' = \Sigma m x^2 + \Sigma m h^2 + \Sigma 2 m x h.$$

But $\qquad \Sigma m x^2 = I; \Sigma m h^2 = M h^2.$

$\therefore \qquad I' = I + M h^2 + 2 h \Sigma m x.$

Now, $\Sigma m x$ is the algebraic sum of the moments of all the particles about $G$.

Since the body is balanced about the centre of mass $G$, $\Sigma m x = 0$.

$\therefore \qquad I' = I + M h^2.$

## 10.5. KINETIC ENERGY OF A ROTATING BODY

Consider a rigid body of mass $M$ rotating with an angular velocity $\omega$ about an axis through $O$ (Fig. 10.6). The body is composed of a large number of particles of masses $m_1, m_2, m_3, \ldots$ situated at distances $r_1, r_2, r_3, \ldots$ from the axis of rotation. Every particle revolves with the same angular velocity $\omega$.

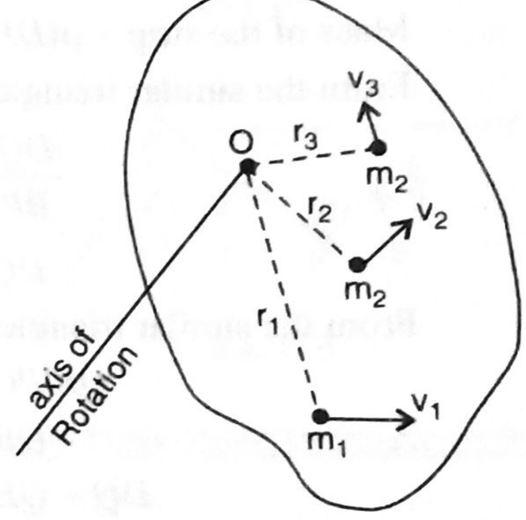


Fig. 10.6

The linear velocity ($v_1$) of the particle of mass $m_1$ is $r_1\omega$. Then

$$\text{K.E of the particle of mass } m_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 r_1^2 \omega^2.$$

Similarly, K.E. of particle of mass $m_2 = \frac{1}{2} m_2 r_2^2 \omega^2$ and so on.

$$\therefore \quad \text{The K.E. of the whole body } E = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \ldots..$$

$$= \frac{1}{2} (\Sigma m r^2)\, \omega^2$$

$$= \frac{1}{2} I \omega^2$$

Here, $I = \Sigma m r^2$, is the moment of inertia of the body about the axis of rotation.

$$\therefore \quad E = \frac{1}{2} I \omega^2$$

If $\omega = 1; I = 2 \times E$

Hence the moment of inertia of a body may also be defined as twice its kinetic energy of rotation, the angular velocity of the body being unity.

Angular momentum, $L = I\omega$

$\therefore$ Kinetic energy of rotation $E = \dfrac{L^2}{2I}$

This is the relation between rotational kinetic energy and angular momentum of a body about the same axis.

## 10.7. TORQUE

Consider a particle at a point $P$ (Fig. 10.8). Let the position vector of $P$ relative to an origin $O$ be $\mathbf{r}$. Let $\mathbf{F}$ be the force acting on the particle. Then the torque $\vec{\tau}$ acting on the particle with respect to the origin $O$ is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \quad ...(1)$$

Torque is a vector quantity. Its magnitude is given by

$$\tau = rF \sin\theta \quad ...(2)$$

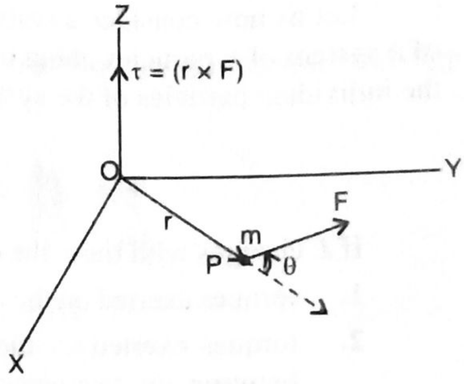


Fig. 10.8

Here, θ is the angle between $\mathbf{r}$ and $\mathbf{F}$. Its direction is normal to the plane formed by $\mathbf{r}$ and $\mathbf{F}$. If one swings $\mathbf{r}$ into $\mathbf{F}$ through the smaller angle between them with the curled fingers of the right hand, the direction of the extended thumb gives the direction of $\vec{\tau}$.

Torque has the dimensions of force times distance, *i.e.*, $ML^2T^{-2}$.

The unit of torque is newton metre (Nm).

## 10.8. ANGULAR MOMENTUM

Consider a particle of mass $m$ and linear momentum $\mathbf{p}$ at a position $\mathbf{r}$ relative to the origin O (Fig. 10.9). The angular momentum $\mathbf{l}$ of the particle with respect to the origin $O$ is defined as

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v})$$

Angular momentum is a vector. Its magnitude is given by

$$l = rp \sin\theta$$

**Here,** θ is the angle between $\mathbf{r}$ and $\mathbf{p}$. Its direction is normal to the plane formed by $\mathbf{r}$ and $\mathbf{p}$. The sense is given by the right-hand rule.

The unit of angular momentum is kg $m^2s^{-1}$ or Js.

For circular motion, $v = r\omega$. The magnitude of $\mathbf{l}$ is $mr^2\omega = I\omega$.

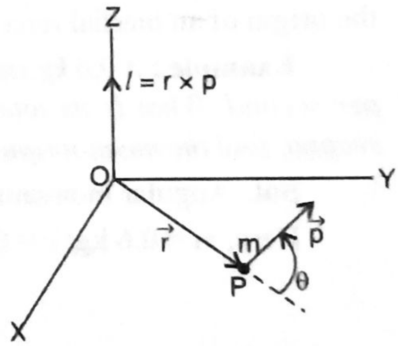


Fig. 10.9

## 10.9. RELATION BETWEEN TORQUE AND ANGULAR MOMENTUM

We have the relation $\mathbf{l} = \mathbf{r} \times \mathbf{p}$

Differentiating this relation with respect to $t$,

$$\frac{d\mathbf{l}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

$$d\mathbf{r}/dt = \mathbf{v} = \text{instantaneous velocity.}$$

Now, $(d\mathbf{r}/dt) \times \mathbf{p} = \mathbf{v} \times (m\mathbf{v}) = 0.$

and $d\mathbf{p}/dt$ = Force $\mathbf{F}$ acting on the particle.

$$\therefore \quad \frac{d\mathbf{l}}{dt} = \mathbf{r} \times \mathbf{F}$$

But, by definition, $\mathbf{r} \times \mathbf{F} = \vec{\tau}$ = torque of $\mathbf{F}$ about $O$.

$$\therefore \quad \frac{d\mathbf{l}}{dt} = \vec{\tau}.$$

**Thus, the time rate** of change of the angular momentum of a particle is equal to the torque **acting on it.**

## 6.10. THE COMPOUND PENDULUM

Any rigid body capable of oscillating freely about a horizontal axis passing through it is a compound pendulum.

**To find the period of oscillation of a compound pendulum :**

Let $O$ be the point of suspension and $G$ the centre of mass (Fig. 6.8). In the equilibrium position, $OG$ is vertical. $OG = h$. Suppose the body is given a small angular displacement about $O$ and let go. The centre of mass $G$ is displaced to $G'$. The body oscillates about the equilibrium position. It can be shown that the motion is simple harmonic. Let $M$ be the mass of the pendulum. The restoring couple due to gravity $= Mgh \sin \theta$. The couple is also equal to $I\,(d^2\theta/dt^2)$ where $I$ = M.I. of the body about the axis of rotation and $(d^2\theta/dt^2)$ = angular acceleration.

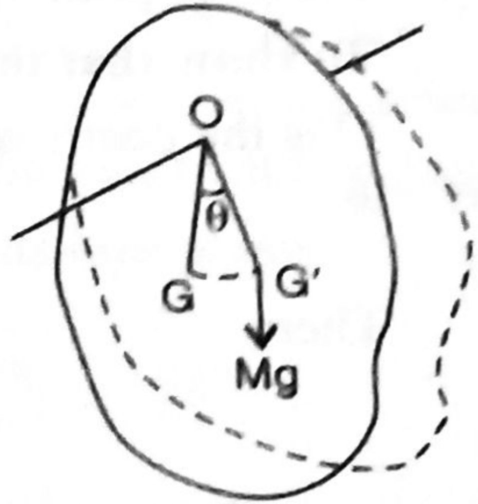


Fig. 6.8

The equation of motion of the body is

$$I\frac{d^2\theta}{dt^2} = -Mgh\sin\theta$$

or $$\frac{d^2\theta}{dt^2} = -\frac{Mgh}{I}\sin\theta = -\frac{Mgh}{I}\theta \quad [\because \sin\theta = \theta \text{ when } \theta \text{ is small}]$$

Now $Mgh/I$ is a constant quantity. Therefore, angular acceleration $(d^2\theta/dt^2)$ is proportional to the angular displacement $\theta$ and the motion of the body is simple harmonic.

$$\text{Period } T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\frac{\theta}{\left(\frac{Mgh}{I}\theta\right)}}$$

$$= 2\pi\sqrt{\frac{I}{Mgh}}$$

$I = I_{cm} + Mh^2$ by the parallel axes theorem

$I = Mk^2 + Mh^2$ [where $k$ = Radius of gyration of the body about an axis passing through cm]

or $I = M(k^2 + h^2)$.

$$\therefore \quad T = 2\pi\sqrt{\frac{M(k^2+h^2)}{Mgh}} \text{ or } T = 2\pi\sqrt{\frac{k^2+h^2}{gh}} \quad ...(1)$$

**Equivalent simple pendulum**

A simple pendulum which oscillates with the same period as the compound pendulum is called the equivalent simple pendulum.

In the case of a simple pendulum of length $L$, the period

$$T = 2\pi\sqrt{\frac{L}{g}} \quad ...(2)$$

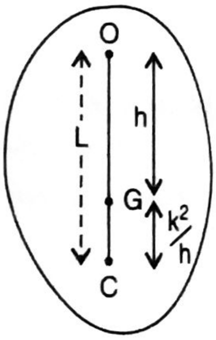


Fig. 6.9

Comparing expressions (1) and (2),

the length of the equivalent simple pendulum $= L = \frac{k^2+h^2}{h}$

**Centre of oscillation :** Produce the line $OG$ to $C$ such that

$$OC = \frac{k^2+h^2}{h} = h + \frac{k^2}{h}$$

Then the point $C$ is called the centre of oscillation (Fig. 6.9).

**To show that the centre of suspension and the centre of oscillation are interchangeable.**

$O$ is the centre of suspension and $C$ is the centre of oscillation of a compound pendulum and $OG = h$.

Then $$OC = L = \frac{k^2+h^2}{h}$$

$$\therefore \quad k^2 = hL - h^2 = h(L-h) = OG \times CG.$$

The symmetry of the expression $k^2 = OG \times CG$ shows that, if the body is suspended about a parallel axis through $C$, $O$ would be the centre of oscillation. Hence the centre of oscillation and suspension are interchangeable.

## Minimum time of oscillation of a compound pendulum

$$T = 2\pi\sqrt{\frac{k^2 + h^2}{gh}}$$

$T$ will be a minimum when, $\frac{d}{dh}\left[\frac{k^2 + h^2}{h}\right] = 0$

i.e., $\frac{d}{dh}\left[h + \frac{k^2}{h}\right] = 0$, i.e., $1 - \frac{k^2}{h^2} = 0$ or $k = h$

Therefore, the period is a minimum when the radius of gyration about a parallel axis through the centre of mass of the body = The depth of the cm below the point of suspension.

## Determination of *g* with compound pendulum

A compound pendulum consists of a heavy uniform metal bar about a metre long. It has a number of holes drilled at regular intervals on either side of the centre of mass $G$ [Fig. 6.10].

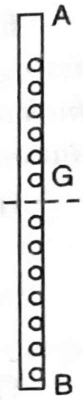


Fig. 6.10

The horizontal knife-edge is passed through the hole near the end $A$. The period of oscillation is determined and the distance of the knife-edge from the end $A$ is measured. The experiment is repeated and the bar is made to oscillate about the knife-edge placed successively in the different holes from $A$ to $B$. In each case the period of oscillation ($T$) and the distance of the position of the knife-edge from the same end $A$ are noted.

A graph is plotted between the period, (on $y$-axis) and the distance from $A$ (on the $x$-axis).

Two curves as shown in Fig. 6.11 are obtained. A horizontal line $PQRS$ is drawn cutting both the curves at points $P, Q, R$ and $S$. $P, Q, R$ and $S$ are then the four points on the bar collinear with the centre of mass having the same period.

$PR = QS = L$, the length of the equivalent simple pendulum. Therefore, if $t$ be its time period given by the ordinate of any one of the points $P, Q, R$ or $S$ we have,

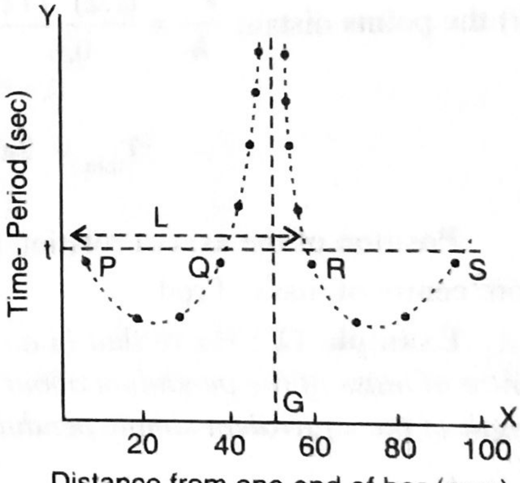


Fig. 6.11

$$t = 2\pi\sqrt{\frac{L}{g}} \text{ or } g = 4\pi^2\frac{L}{t^2}$$

Knowing $L$ and $t$, we can calculate the value of $g$ at the given place.

**Example 10 :** *A compound pendulum is formed by suspending a heavy ring of radius 4.905 metres. What is the length of the string for the minimum period ? Calculate this period.*

Since the ring is large and heavy, the weight of the string may be neglected so that the centre of mass of the compound pendulum will be the same as that of the ring, *i.e.*, the centre of the ring.

Let $m$ and $R$ be the mass and radius of the ring. If the ring oscillates in its own plane,

M.I. of the ring about an axis through its centre and $\perp r$ to its plane $\Big\} = I = mR^2 = mk^2$

where $k$ is the radius of gyration of the ring about an axis through its centre and perpendicular to its plane.

$\therefore \quad k = R,$

The period is minimum when $k = h$, where $h$ = the distance of the point of suspension from the centre of mass of the ring. This gives $h = R$.

*i.e.*, the length of the string is zero and the ring is suspended at a point on its periphery.

We have, $T = 2\pi\sqrt{\dfrac{k^2+h^2}{gh}}$. $T$ is minimum when $k = h$

$$\therefore \quad T_{min} = 2\pi\sqrt{\frac{k^2+k^2}{kg}} = 2\pi\sqrt{\frac{2k}{g}} = 2\pi\sqrt{\frac{2R}{g}} \qquad (\because k = R)$$

Here, $R = 4.905$ m, $g = 9.81$ ms$^{-2}$.

Hence $T_{min} = 2\pi\sqrt{\dfrac{2\times 4.905}{9.81}} = 6.282$ s.

**Example 11 :** *A uniform rod of length 1.2 m oscillates about a horizontal axis of rotation passing through one end. Find the period of oscillation. Find the positions of the other points about which the period is same. Also calculate the minimum period possible and the position of the axis of rotation for obtaining the minimum period.*

Here, $k^2 = l^2/12 = (1.2)^2/12$ ; $h = l/2 = 0.6$ m.

$$\therefore \quad T = 2\pi\sqrt{\frac{k^2+h^2}{gh}} = 2\pi\sqrt{\frac{(1.2)^2/12+(0.6)^2}{9.8\times 0.6}} = 1.794 \text{ s}$$

The position of other points, about which the period is same : (*i*) the other end of the rod ; (*ii*) the points distant $\dfrac{k^2}{h} = \dfrac{(1.2)^2/12}{0.6} = 0.2$ m from the centre and on either side.

$$T_{min} = 2\pi\sqrt{\frac{2k}{g}} = 2\pi\sqrt{\frac{2\times(1.2/\sqrt{12})}{9.8}} = 1.671 \text{ s}$$

Position of the axis of rotation for obtaining the minimum period = $k = 1.2/\sqrt{12} = 0.346$ m from centre of mass of rod.

**Example 12 :** *Prove that in a compound pendulum, there are four points collinear with the centre of mass of the pendulum about which it has the same period of oscillation. Hence obtain the length of the equivalent simple pendulum.*

$$T = 2\pi\sqrt{\frac{k^2+h^2}{g\,.\,h}}; \text{ Squaring, } T^2 = \frac{4\pi^2(k^2+h^2)}{g\,.\,h}$$

Rearranging, $h^2 - \dfrac{g\,.\,T^2}{4\pi^2}h + k^2 = 0$

This is a quadratic equation in $h$ and gives two values of $h$.

$$h_1 = \frac{g\,.\,T^2}{8\pi^2} + \sqrt{\frac{g^2\,T^4}{64\pi^4} - k^2}$$

and

$$h_2 = \frac{g\,.\,T^2}{8\pi^2} - \sqrt{\frac{g^2T^4}{64\pi^4} - k^2}$$

Thus there are two points on either side of *cm* lying at distances $h_1$ and $h_2$ respectively about which the time period of the compound pendulum has the same value. Hence in all there are four points collinear with the *cm* about which the periods are equal.

$$h_1 + h_2 = g \cdot T^2/(4\pi^2) = \text{Length of the equivalent simple pendulum.}$$

**Example 13 :** *A uniform circular disc of radius R oscillates in a vertical plane about a horizontal axis. Find the distance of the axis of rotation from the centre for which the period is minimum. What is the value of this period ?*

We have, $$T = 2\pi\sqrt{(k^2 + h^2)/(g \cdot h)}$$

$T$ is minimum when $h = k$.

$$\therefore \quad T_{min} = 2\pi\sqrt{(k^2 + k^2)/(kg)} = 2\pi\sqrt{2k/g}$$

M.I. of a disc about an axis perpendicular to its plane and passing through its centre $\Big\} = I = Mk^2 = \frac{1}{2}MR^2$

where $M$ is the mass of the disc and $R$, its radius.

$$\therefore \quad k = R/\sqrt{2}.$$

Thus, $T$ is minimum when the distance of the axis of rotation from the centre is $R/\sqrt{2}$.

$$T_{min} = 2\pi\sqrt{\frac{2R/\sqrt{2}}{g}} = 2\pi\sqrt{\frac{R\sqrt{2}}{g}} = 2\pi\sqrt{\frac{1.414\,R}{g}}$$

**Example 14 :** *A uniform circular disc of 0.2 m radius oscillates in its own plane about a point on its circumference. Calculate the period of oscillation.*

We have, $$T = 2\pi\sqrt{(k^2 + h^2)/(g \cdot h)}$$

Here, $I = Mk^2 = MR^2/2$ or $k^2 = R^2/2$ ; $h = R$

$$\therefore \quad T = 2\pi\sqrt{\frac{(R^2/2) + R^2}{g \cdot R}} = 2\pi\sqrt{\frac{3R}{2g}} = 2\pi\sqrt{\frac{3 \times 0.2}{2 \times 9.8}}$$

$$= 1.099 \text{ s}$$

## 9.5. BIFILAR PENDULUM-PARALLEL THREADS

When a heavy uniform rod is suspended horizontally by means of two equal vertical threads at equal distances from the centre of gravity of the rod, the arrangement is called a bifilar suspension. The suspension may be made by means of two parallel or two nonparallel threads. We consider here suspension with parallel threads.

Let $AB$ represent the equilibrium position of the rod of mass $m$. The weight of the rod $mg$ acts vertically downwards at its centre of gravity $O$ (Fig. 9.7). The rod is suspended by two equal threads ($PA$ and $QB$) of length $l$ and separated by a distance $2d$. Suppose now the rod is displaced into the position $A'B'$ through a small angle $\theta$, about a vertical axis through $O$.

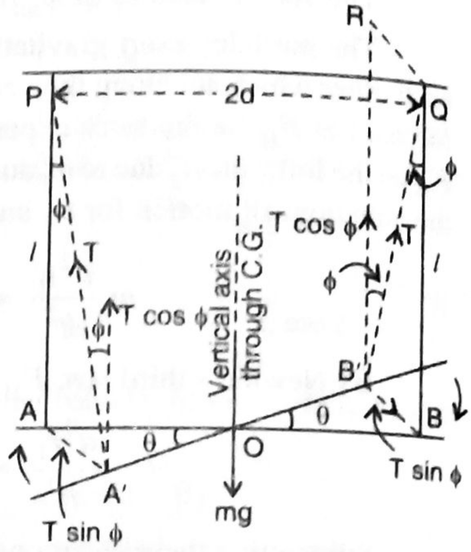


Fig. 9.7

The threads now take up the position $PA'$ and $QB'$ at an angle $\theta$ with their original positions.

Let $T$ be the tension in each thread, acting upwards along the thread. $T$ may be resolved into two components, one vertically and the other horizontally. The vertical components are each of magnitude $T\cos\phi$, which balance the weight of the rod. Hence, $2T\cos\phi = mg$ or $T\cos\phi = mg/2$.

Now from the figure, $d\theta = l\phi$ or $\phi = \dfrac{d\theta}{l}$

$$\cos\phi = \cos\left(\frac{d\theta}{l}\right) = 1 \qquad \text{since } \theta \text{ is small.}$$

$$\therefore \qquad T = mg/2.$$

Now the horizontal component of the tension ($T\sin\phi$) is along $B'B$ and $A'A$. The two horizontal components of the tension are equal, opposite and parallel. Hence they constitute a couple, tending to bring the rod back into its original position. The two forces act at right angles to the axis of the rod. Hence moment of the restoring couple = $(T\sin\phi)\,2d = T\phi\,2d$. $(\because \sin\phi = \phi)$

$$= T\left(\frac{\theta d}{l}\right)2d = \frac{mg}{2}\frac{\theta d}{l}2d = \frac{mgd^2}{l}\theta.$$

Let $I$ be the M.I., of the rod about its axis of rotation and $d^2\theta/dt^2$, its angular acceleration. Hence the equation of motion of the rod is given by

$$I\frac{d^2\theta}{dt^2} = \frac{-mgd^2}{l}\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = \frac{-mgd^2}{Il}\theta$$

Clearly, $\dfrac{d^2\theta}{dt^2} \propto \theta$ since $\dfrac{mgd^2}{Il}$ is a constant quantity.

Hence the rod executes S.H.M., and its time period is

$$t = \frac{2\pi}{\sqrt{mgd^2/(Il)}} = 2\pi\sqrt{\frac{Il}{mgd^2}}$$

Now $I = mk^2$ where $k$ is the radius of gyration of the rod about the axis of rotation. Hence

$$t = 2\pi\frac{k}{d}\sqrt{\frac{l}{g}}$$

or

$$g = 4\pi^2k^2l/(d^2t^2).$$

Thus $g$ at the given place is determined.